※請在答案卡內作答

•本測驗試題為多選題(答案可能有一個或多個)，請選出所有正確或最適當的答案，並請用 2B 鉛筆作答於答案卡。

•共二十題，每題五分。每題 ABCDE 每一選項單獨計分；每一選項的個別分數為一分，答錯倒扣一分。

Notation: In the following questions, boldface letters such as **a**, **b**, etc. denote columns vectors of proper length; boldface letters such as **A**, **B** etc. denote matrices of proper size; $\mathbf{A}^T$ means the transpose of matrix **A**, and $\mathbf{A}^H$ is the Hermitian transpose (a.k.a. conjugate transpose) of **A**. $\mathbf{I}_n$ is the $(n \times n)$ identity matrix. $\|\mathbf{a}\|_2$ means the Euclidean norm of vector **a**. $\mathbb{R}$ is the usual set of all real numbers; $\mathbb{C}$ is the usual set of all complex numbers. $\mathbb{R}^{m \times n}$ means the set of all $(m \times n)$ real-valued matrix, and similarly for $\mathbb{C}^{m \times n}$. The symbol $\mathcal{L}\{y(t)\}$ denotes the Laplace transform of $y(t)$.

一、Let **A** and **B** be real-valued matrices. Which of the following statement is/are true?

(A). $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.

(B). If $\mathbf{AB} = \mathbf{B}$ then $\mathbf{A} = \mathbf{I}_n$.

(C). $\mathbf{A}^{-1}$ can be asymmetric if **A** is symmetric.

(D). If **A** and **B** are invertible, then **A**+**B** is invertible.

(E). None of the above is true.

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二、Consider the vector space S consisting of all degree-2 polynomials with real coefficients, i.e., polynomials in the form of $c_0 + c_1t + c_2t^2$. Define the inner product between two vectors as $< c_0 + c_1t + c_2t^2, d_0 + d_1t + d_2t^2> = c_0d_0 + c_1d_1 + c_2d_2$. Which of the following statement is/are true?

(A). The dimension of S is 2.

(B). The polynomials $1 + t$, t and $1 + t^2$ are linearly independent.

(C). The set $\{1 + t, t, 1 + t^2\}$ can be a basis for S.

(D). The two polynomials $1 - 2t + t^2$, and $-1 + 2t - t^2$ are orthogonal to each other.

(E). None of the above is true.

三、Let $\mathbf{A}$ be an $m \times n$ real-valued matrix of rank r. and $\mathbf{b}$ be an $m \times 1$ real-valued vector. Which of the following statement is/are true?

(A). The equation $\mathbf{Ax} = \mathbf{b}$ has non-trivial solutions if $n > r > 1$ and $\mathbf{b}$ is all-zero.

(B). The equation $\mathbf{Ax} = \mathbf{b}$ has solutions if $\mathbf{b}$ belongs to the column space of $\mathbf{A}$.

(C). The equation $\mathbf{Ax} = \mathbf{b}$ has only one solution when $r = m$.

(D). The dimension of the nullspace of $\mathbf{A}$ is 0 if the dimension of the row space of $\mathbf{A}$ is r.

(E). None of the above is true.

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四、Let **A** be a 6×6 block-diagonal matrix $\begin{bmatrix} \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 3\mathbf{B} \end{bmatrix}$, where $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and **0**'s are 2×2 zero matrices. Furthermore, the vector $\mathbf{b} = [1, 2, 3, 4, 5, 6]^T$. Which of the following statement is/are true?

(A). $\mathbf{A}^T = -\mathbf{A}$.

(B). $\mathbf{A}^{-1}$ is also a block-diagonal matrix.

(C). The **LU** factorization of **A** will produce a diagonal matrix **U**.

(D). Solve the equation $\mathbf{Ax} = \mathbf{b}$, then the solution's last entry is 11/6.

(E). None of the above is true.

五、Consider the vector space $\mathbb{R}^4$. The vector $\mathbf{u} = [1, 1, 1, 1]^T$, $\mathbf{v} = [1, 1, -1, -1]^T$, and $\mathbf{w} = [1, -1, 1, -1]^T$. Which of the following statement is/are true?

(A). The matrix $\mathbf{P} = \mathbf{u}\mathbf{u}^T$. Then $\mathbf{Px}$ produces the orthogonal projection of a vector **x** onto the line spanned by **u**.

(B). The matrix $\mathbf{Q} = \mathbf{v}\mathbf{v}^T$. Then **PQ** is an all-zero matrix.

(C). The matrix $\mathbf{A} = [\mathbf{u}, \mathbf{v}, \mathbf{w}]$ has orthogonal columns.

(D). Continue form part (C). If a vector **b** can be spanned by the columns of **A**, i.e., $\mathbf{b} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}$, then $c_1 = \mathbf{u}^T\mathbf{b}/4$.

(E). None of the above is true.

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六、Let $\mathbf{A}=\mathbf{A}^T$ and assume its eigenvalue can be any element in $\mathbb{R}$. The eigenspace is the same as

(A). Column space only

(B). Row and column space only

(C). Row and nullspace only

(D). All of the fundamental subspaces

(E). None of the above is true.

七、Let $\mathbf{A}^H\mathbf{A} = \mathbf{A}\mathbf{A}^H = \mathbf{I}_n$. λ_i denotes the ith eigenvalue of $\mathbf{A}$, and $|.|$ denote the magnitude operator. Which of the following statement is/are true?

(A). λ_i equals to its pivots, $\forall i$.

(B). λ_i is real, $\forall i$.

(C). $|\lambda_i|$ equals to $\sqrt{n}$ only, $\forall i$.

(D). $|\lambda_i|$ equals to 1 only, $\forall i$.

(E). None of the above is true.

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八、If $det(\mathbf{A}) = det(2\mathbf{A})$, then

(A). $\mathbf{A}$ is the identity matrix

(B). $\mathbf{A}$ has a zero row

(C). $\mathbf{A}$ is non-invertible

(D). $\mathbf{A} = \mathbf{A}^T$

(E). None of the above is true.

九、Let T be a differential operator $T(f) = df/dx$. Given the basis $\{1, x, \sin(3x), \cos(3x)\}$. What is the matrix $\mathbf{A}$ for the operator T?

(A). $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & -3 & 0 \end{bmatrix}$

(B). $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

(C). $\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & -2 & 0 \end{bmatrix}$

(D). $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -3 & 0 \end{bmatrix}$

(E). None of the above is true.

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十、Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \neq \mathbf{0}_{2\times1}$. The minimum of $\frac{\mathbf{x}^T\begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$ with respect to $\mathbf{x}$ is

(A). $2\|\mathbf{x}\|_2$

(B). $3x+5y$

(C). -4

(D). -2

(E). None of the above is true.

十一、 Solving the first-order differential equation $xy'(x) = y(x) + \sqrt{x^2 + y^2}$ with the initial condition $y(3) = 4$. Which of the following statements is/are true?

(A). It is a nonlinear differential equation.

(B). The particular solution is $y + \sqrt{x^2 + y^2} = x^2$.

(C). The particular solution is $y = \frac{1}{2}(x^2 - 1)$.

(D). $y(0) = 0$.

(E). None of the above is true.

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十二、 Solve the following nonhomogeneous second-order differential equation $(1-x)y''(x) + xy'(x) - y(x) = 2(x-1)^2 e^{-x}$ by the variational method, *i.e.* $y_P(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, where $y_1(x)$ and $y_2(x)$ are solutions of the associated homogeneous equation given by x and e^x, respectively. Which of the following statements is/are true?

(A). $u_1'(x) = -2(x-1)e^{-x}$.

(B). $u_1'(x) = 2e^{-x}$.

(C). $u_2'(x) = 2x(x-1)e^{-2x}$.

(D). $u_2'(x) = -2xe^{-2x}$.

(E). None of the above is true.

十三、 The linear system $\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 10 & 0 & 0 \\ 10 & 6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$ can be solved by finding the eigenvalues of its coefficient matrix. Regarding these eigenvalues, which of the following statements is/are true?

(A). All eigenvalues are real.

(B). All eigenvalues are pure imaginary.

(C). The 4 eigenvalues of -2, 2, -4 and 4.

(D). The 4 eigenvalues are *-2i*, *2i*, *-4i*, and *4i*

(E). None of the above is true.

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十四、 Continued from Problem 十三, the particular solution that satisfies the initial conditions $x_1(0) = x_2(0) = 1, x_3(0) = -6, x_4(0) = -2$ is

(A). $x_1(t) = e^{-4t} - \sin(2t)$

(B). $x_2(t) = e^{-4t} + \sin(2t)$

(C). $x_3(t) = -4e^{-4t} - 2\cos(2t)$

(D). $x_4(t) = -4e^{-4t} + 2\cos(2t)$

(E). None of the above is true.

十五、 Use the Laplace transform and the convolution theorem to solve the following integro-differential equation, where

$$y'(t) = \int_0^t y(u)\cos(t-u)\,du, \quad y(0) = 1.$$

Which of the following statements is/are true?

(A). $\mathcal{L}\{y(t)\} = \frac{1}{s} + \frac{s}{s^2+2}$.

(B). $\mathcal{L}\{y(t)\} = \frac{1}{s} + \frac{s}{s^2+1}$.

(C). $y(t) = \frac{1}{2}e^t + \frac{1}{2}\cos(2t)$.

(D). $y'(t) = e^t - 2\sin(2t)$.

(E). None of the above is true.

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十六、 Consider the following differential equation:

$$(x^2-4)y''+3xy'+y=0$$

with $y(0)=4$ and $y'(0)=1$. The series solution is $y(x)=\sum_{n=0}^{\infty}c_n x^n$. Which of the following statements is/are true?

(A). The radius of convergence for the series solution is 2.

(B). There is no singular point.

(C). There are two linearly independent solutions.

(D). $c_4=\frac{3}{32}$.

(E). None of the above is true.

十七、 Consider the following differential equation:

$$xy''+2y'+xy=0$$

with $y(0)=1$. Which of the following statements is/are true?

(A). The radius of convergence for the series solution is 1.

(B). There is no singular point.

(C). There are two linearly independent solutions.

(D). $y(\pi)=0$.

(E). None of the above is true.

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十八、 Consider $F(t)$ is a period-4 function with $F(t) = 5t$ for $-2 < t < 2$.

Find its Fourier series $f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L} \right)$. Which of the following statements is/are true?

(A). $L = 2$.

(B). $a_0 = 0$.

(C). $a_1 = \frac{10}{\pi}$.

(D). $b_2 = -\frac{10}{\pi}$.

(E). None of the above is true.

十九、 Continued from Problem 十八, the steady solution of

$$x'' + 10x = F(t)$$

is $x(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{D}$. Which of the following statements is/are true?

(A). $D = 4$.

(B). The dominant term is the $n = 2$ term.

(C). The dominant frequency is $\sqrt{10}$.

(D). The dominant frequency is π.

(E). None of the above is true.

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二十、 Consider the endpoint problem,

$$y'' + \lambda y = 0, \quad y(0) = y(L) = 0, L > 0.$$

Determine the eigenvalues and associated eigenfunctions. Which of the following statements is/are true?

(A). $Ax + B$ is a non-trivial solution, where A and B are constants.

(B). All eigenvalues are non-negative.

(C). There is at least one eigenvalue shared by two linearly independent eigenfunctions.

(D). 0 is one of the eigenvalues.

(E). None of the above is true.