- 本測驗試題為多選題（答案可能有一個或多個），請選出所有正確或最適當的答案，並請將答案用2B鉛筆填於答案卡。
- 共二十題，每題五分。每題ABCDE選項單獨計分；每一選項個別分數為一分，答錯倒扣一分，倒扣至本測驗試題零分為止。

Notation: In the following problems, $\mathbb{R}$ is the usual set of all real numbers. We will use underlined letters such as $\underline{a} \in \mathbb{R}^n$ to denote a real, column vector $\underline{a}$ of length n and similarly will use boldface letters such as $\mathbf{A} \in \mathbb{R}^{m\times n}$ to denote a real matrix $\mathbf{A}$ of size $m \times n$. $\underline{0}$ is the all-zero column vector of proper length. $\mathbf{A}^\top$ is the transpose of matrix $\mathbf{A}$. rank($\mathbf{A}$) denotes the rank of matrix $\mathbf{A}$. $\mathbf{I}_n$ is the $n \times n$ identity matrix. det($\mathbf{A}$) and tr($\mathbf{A}$) are respectively the determinant and trace of square matrix $\mathbf{A}$. Unless otherwise stated, all vector spaces and linear combinations are over field $\mathbb{R}$ and the orthogonality is with respect to the usual Euclidean inner product. Primes of functions of one variable denotes the derivatives with respective the variable, for instance, $y'(x) = \frac{d}{dx}y(x)$.

1. Consider the vectors that end at the vertices of the unit cube in the $\mathbb{R}^{10}$ space, i.e. the vectors in the form of $[x_1, x_2, \cdots, x_{10}]^\top$ in which x_i, $i = 1, 2, \ldots, 10$, is either zero or one but not all x_i's are zero simultaneously. We then select from these vectors to form a set such that vectors in the set are mutually orthogonal to each other. At most how many vectors can the set have?

   (A) 2.

   (B) 10.

   (C) 10^2.

   (D) 2^{10}.

   (E) None of the above is true.

2. Which of the following statements is/are true?

(A) A real matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

(B) For a matrix

$$\mathbf{A} = \begin{bmatrix} 0 & \times & 0 \\ \times & \times & \times \\ 0 & \times & 0 \end{bmatrix}$$

regardless of the values of the five elements marked by $\times$'s (they can be different), $\mathbf{A}$ cannot have 1 as its eigenvalue.

(C) If a 3×3 matrix has three eigenvalues 0, 1, 2, then its rank is 3.

(D) The rotation matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

does not have eigenvectors in $\mathbb{R}^2$, unless θ is an integral multiple of π.

(E) None of the above is true.

3. Consider the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 0.7 & -0.3 \\ -0.3 & 0.7 \end{bmatrix}$$

Which of the following statements is/are true?

(A) $\mathbf{A}$ is diagonalizable.

(B) $\mathbf{A}$ is positive definite.

(C) $\mathbf{A}$ and $\mathbf{A}^2$ have the same eigenvectors.

(D)

$$\lim_{k\to\infty} \mathbf{A}^k = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

(E) None of the above is true.

4. Which of the following statements is/are true?

(A) For a nonzero matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, let $\mathbf{P}$ be a matrix that orthogonally projects vectors in $\mathbb{R}^n$ onto the column space of $\mathbf{A}$. Then $\mathbf{I}_n - \mathbf{P}$ projects onto the right null space of $\mathbf{A}$.

(B) If $\mathbf{A} \in \mathbb{R}^{3\times 2}$ has two orthonormal columns, then $\mathbf{A}^\top \mathbf{A} = \mathbf{I}_2$.

(C) The orthogonal projection of vectors in $\mathbb{R}^3$ onto the column space of $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}$ can be obtained by multiplying the vector with the projection matrix $\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(D) Three orthogonal vectors in $\mathbb{R}^4$ can be constructed by the Gram-Schmidt procedure from the vectors $[1,-1,0,0]^\top$, $[0,1,-1,0]^\top$ and $[1,0,0,-1]^\top$.

(E) None of the above is true.

5. Let $\mathcal{P}_n$ be the set of polynomials in x with degree at most n and with real-valued coefficients. For any $u(x) = c_0 + c_1 x + \cdots + c_n x^n$ and $v(x) = d_0 + d_1 x + \cdots + d_n x^n$ in $\mathcal{P}_n$, define the following inner product

$$\langle u(x), v(x) \rangle = c_0 d_0 + c_1 d_1 + \cdots + c_n d_n.$$

Note that $\mathcal{P}_n$ can be regarded as a normed vector space. Which of the following statements is/are true?

(A) $\mathcal{P}_2$ is a subspace of $\mathcal{P}_3$.

(B) The norm of $1 + x + x^2$ is 3 in $\mathcal{P}_2$.

(C) The set $\{1, x, x^2\}$ can be an orthonormal basis for $\mathcal{P}_2$.

(D) The operation $u(x) \mapsto u(x)/x$ maps elements from $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$.

(E) None of the above is true.

6. Which of the following statements about vector space is/are true?

   (A) A vector is an arrow in three-dimensional space.

   (B) A subset $\mathcal{H}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if the zero vector is in $\mathcal{H}$.

   (C) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

   (D) Let $\mathcal{P}_n$ be the set of polynomials $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ with degree at most n and coefficients $a_0, \ldots, a_n \in \mathbb{R}$. Then, the set of polynomials with degree at most 3 and integer coefficients, is a subspace of $\mathcal{P}_5$.

   (E) None of the above is true.

7. Which of the following statements about the multiplicative inverse of square matrices is/are true?

   (A) The columns of an invertible matrix in $\mathbb{R}^{n\times n}$ form a basis for $\mathbb{R}^n$.

   (B) Let $\mathbf{A}$ and $\mathbf{P}$ be square matrices of the same size. Suppose that $\mathbf{P}$ is invertible with its multiplicative inverse $\mathbf{P}^{-1}$. Then $\det(\mathbf{PAP}^{-1}) = \det(\mathbf{A})$.

   (C) Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$ be matrices and suppose that matrices $\mathbf{A}$, $\mathbf{C}$ and $(\mathbf{A} - \mathbf{AC})$ are invertible with multiplicative inverses respectively $\mathbf{A}^{-1}$, $\mathbf{C}^{-1}$ and $(\mathbf{A} - \mathbf{AC})^{-1}$. If $(\mathbf{A} - \mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{B}$, then $\mathbf{B}$ must be invertible.

   (D) The matrix $\begin{bmatrix} 2 & 2 & 8 & -2 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$ is invertible.

   (E) None of the above is true.

8. Which of the following statements about determinant is/are true?

(A) The determinant of a triangular matrix is the sum of the entries on the main diagonal.

(B)

$$\det\left(\begin{bmatrix} 1 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ -8 & 5 & -6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ -4 & 2 & 3 & 2 & 0 \end{bmatrix}\right) = 6$$

(C) Let $\mathbf{U} \in \mathbb{R}^{n\times n}$ be such that $\mathbf{U}^\top \mathbf{U} = \mathbf{I}_n$. Then $\det(\mathbf{U})$ equals either 1 or -1.

(D) Replacing a row of a square matrix by the sum of itself and a multiple of another row changes the determinant of the matrix.

(E) None of the above is true.

9. Which of the following statements about linear independence is/are true?

(A) If $\mathcal{S}$ is a set of linearly dependent vectors, then each vector in $\mathcal{S}$ is a linear combination of other vectors in $\mathcal{S}$.

(B) The columns of matrix $\begin{bmatrix} 0 & 4 & -3 & 1 \\ 1 & -7 & 5 & -2 \\ 5 & -5 & 7 & -4 \end{bmatrix}$ form a linearly independent set in $\mathbb{R}^3$.

(C) If the matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ satisfies that for any vector $\underline{b} \in \mathbb{R}^m$ the equation $\mathbf{A}\underline{x} = \underline{b}$ has at most one solution to the unknown $\underline{x}$, then the columns of $\mathbf{A}$ must be linearly independent.

(D) If the columns of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ span $\mathbb{R}^n$, then they are linearly independent.

(E) None of the above is true.

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10. Which of the following statements about linear transformation is/are true?

(A) The transformation $T(x, y) = (3x - 2y, x + 3, 6y)$ is not linear.

(B) A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if for each vector $\underline{b} \in \mathbb{R}^m$ there exists at most one vector $\underline{x} \in \mathbb{R}^n$ such that $T(\underline{x}) = \underline{b}$.

(C) If $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a matrix with $m > n$, then the transformation $T(\underline{x}) = \mathbf{A}\underline{x}$ cannot be one-to-one.

(D) An affine transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ in the form of $T(\underline{x}) = \mathbf{A}\underline{x} + \underline{b}$ with $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\underline{b} \in \mathbb{R}^m$ is not a linear transformation when $\underline{b} \neq \underline{0}$.

(E) None of the above is true.

11. The Bernoulli equation

$$2xy'(x) + (y(x))^3\, xe^{-2x} = 2xy(x)$$

with the condition $y(1) = e$ can be solved by the substitution $v(x) = (y(x))^{-2}$. Which of the following statements is/are true?

(A) $y(x) = xe^x$

(B) $v(x) = xe^{-2x}$

(C) $x(y(x))^2 = e^{2x}$

(D) The differential equation after substitution is a linear differential equation for the new variable v.

(E) None of the above is true.

12. For the homogeneous second order linear differential equation

$$4x^2y''(x) - 4xy'(x) + 3y(x) = 0,$$

if given one solution $y_1(x) = \sqrt{x}$, the other linearly independent solution $y_2(x)$ can then be derived by setting $y(x)$ equal to $y_2(x) = v(x)y_1(x)$ and solving for $v(x)$. Which of the following statements is/are true?

(A) $v''(x) = 0$

(B) $v'(x) = 0$

(C) $v'(x) = \frac{1}{x^2}$

(D) $y(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ is a possible solution.

(E) None of the above is true.

13. Continue from Problem 12. Solve the non-homogeneous second order linear differential equation

$$4x^2y''(x) - 4xy'(x) + 3y(x) = 8x^{\frac{4}{3}}$$

by the variation of parameters, i.e., by setting the particular solution as

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are from Problem 12. Which of the following statements is/are true?

(A) $u_1(x) = 12x^{\frac{5}{6}}$

(B) $u_1(x) = -\frac{12}{5}x^{\frac{5}{6}}$

(C) $u_1'(x) = -2x^{-\frac{1}{6}}$

(D) $y_p(x) = 72x^{\frac{4}{3}}$

(E) None of the above is true.

14. The second order system

$$\begin{bmatrix} x''(t) \\ y''(t) \\ z''(t) \end{bmatrix} = \begin{bmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

can be transformed into an equivalent first order system

$$\begin{bmatrix} x_1'(t) \\ y_1'(t) \\ z_1'(t) \\ x_2'(t) \\ y_2'(t) \\ z_2'(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1(t) \\ y_1(t) \\ z_1(t) \\ x_2(t) \\ y_2(t) \\ z_2(t) \end{bmatrix}$$

by introducing the functions

$$x_1(t) = x(t), \quad x_2(t) = x'(t),$$
$$y_1(t) = y(t), \quad y_2(t) = y'(t),$$
$$z_1(t) = z(t), \quad z_2(t) = z'(t).$$

Which of the following statements is/are true?

(A)

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 0 & 4 & -4 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

(B)

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -4 & 4 & 0 & 0 & 0 & 0 \\ 6 & -12 & 6 & 0 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 & 0 \end{bmatrix}$$

(C) 4 is an eigenvalue of $\mathbf{A}$

(D) 0 is an eigenvalue of $\mathbf{A}$

(E) None of the above is true.

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15. Continue from Problem 14. Find the particular solution of the second order system with initial conditions $x(0) = y(0) = z(0) = 0$ and $x'(0) = y'(0) = z'(0) = 12$. Which of the following statements is/are true?

(A) $x(t) = 12t$

(B) $y(t) = 12t - 2\sin(2t) + \sin(4t)$

(C) $z(t) = 12t + 2\sin(2t) - \sin(4t)$

(D) $x(t) = y(t) = z(t)$

(E) None of the above is true.

16. For the following second order differential equation

$$2y''(t) - \frac{1}{t(t-1)}y'(t) + \frac{1}{(t-1)^2}y(t) = 0$$

let $y_1(t) = \sum_{n=0}^{\infty} a_n(t-t_0)^{r_1+n}$ and $y_2(t) = \sum_{n=0}^{\infty} b_n(t-t_0)^{r_2+n}$ be two linearly independent Frobenius series solutions for $y(t)$ at $t = t_0$ for some $t_0 \in \mathbb{R}$. Assuming $r_1 \geq r_2$, which of the following statements is/are true?

(A) $r_1 = 0$ for $t_0 = 0$

(B) $r_2 = -\frac{1}{2}$ for $t_0 = 0$

(C) $r_1 = 1$ for $t_0 = 1$

(D) $r_2 = \frac{1}{2}$ for $t_0 = 1$

(E) None of the above is true.

17. Consider the following differential equation defined for all $t \in \mathbb{R}$

$$y'(t) - 4y(t) + 4\int_0^t \tau e^{\tau} y(t-\tau)\, d\tau = 9$$

subject to the condition $y(0) = 3$. Which of the following statements is/are true regarding the solution $y(t)$ and its power series representation $y(t) = \sum_{n=0}^{\infty} y_n t^n$?

(A) The radius of convergence of the power series equals 5.

(B) $y_1 = 21$

(C) $y_2 = 42$

(D) $y\left(-\frac{1}{2}\right) = \frac{3}{2}$

(E) None of the above is true.

18. Continue from Problem 17. Let $Y(s)$ be the unilateral Laplace transform of $y(t)$ for $t \geq 0$. Which of the following statements is/are true?

(A) $Y(s)$ exists for $\mathrm{Re}\{s\} > 2$

(B) $Y(3) = \frac{137}{9}$

(C) $Y(4) = \frac{179}{16}$

(D) $Y(5) = \frac{96}{25}$

(E) None of the above is true.

19. Use the method of separation of variables to solve the following partial differential equation for the function $g(x, y)$

$$\frac{\partial^2}{\partial x^2} g(x, y) = \frac{\partial^2}{\partial y^2} g(x, y)$$

defined for $0 \le x \le 3$ and $-3 \le y \le 3$, subject to conditions

$$g(0, y) = g(3, y) = 0$$

$$g(x, 0) = x(3 - x)$$

$$g(x, y) = g(x, -y)$$

With the following Fourier series

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \cos\left(\frac{2\pi n x}{3}\right) + b_n \sin\left(\frac{2\pi n x}{3}\right) \right]$$

that is a periodic function in x with period equal to 3 and that equals $g(x, 0)$ for $x \in (0, 3)$, which of the following statements is/are true regarding the values of a_n and b_n?

(A) $a_4 = \frac{8}{3\pi^2}$

(B) $b_4 = -\frac{9}{16\pi^2}$

(C) $a_1 - b_1 = -\frac{9}{\pi^2}$

(D) $a_2 + b_2 = -\frac{9}{4\pi^2}$

(E) None of the above is true.

20. Continue from Problem 19. Which of the following statements is/are true regarding the solution $g(x, y)$?

(A) $g(\frac{1}{2}, \frac{1}{2}) = \frac{5}{8}$

(B) $g(1, 1) = 1$

(C) $g(\frac{3}{2}, \frac{3}{2}) = 0$

(D) $g(2, -2) = 2e^{-2}$

(E) None of the above is true.