- 本測驗試題為多選題（答案可能有一個或多個），請選出所有正確或最適當的答案，並請將答案用2B鉛筆填於答案卡。
- 共二十題，每題五分。每題ABCDE選項單獨計分；每一選項個別分數為一分，答錯倒扣一分，倒扣至本測驗試題零分為止。

Notation: In the following problems, $\mathbb{R}$ is the usual set of all real numbers. We will use underlined letters such as $\underline{a} \in \mathbb{R}^n$ to denote a real, column vector $\underline{a}$ of length n. $\|\underline{a}\|$ means the Frobenius norm of vector $\underline{a}$, and $\underline{0}$ is the all-zero column vector of proper length. We will use boldface letters such as $\mathbf{A} \in \mathbb{R}^{m\times n}$ to denote a real matrix $\mathbf{A}$ of size $m \times n$, and we will write $\mathbf{A} = [a_{i,j}] \in \mathbb{R}^{m\times n}$, where $a_{i,j}$ is the (i,j)-th entry of $\mathbf{A}$ with subindices $i = 1,\ldots,m$, and $j = 1,\ldots,n$. $\mathbf{A}^\top$ is the transpose of matrix $\mathbf{A}$. rank($\mathbf{A}$) denotes the rank of matrix $\mathbf{A}$. $\mathbf{I}_n$ is the $n \times n$ identity matrix. det($\mathbf{A}$) and tr($\mathbf{A}$) are respectively the determinant and trace of square matrix $\mathbf{A}$. Unless otherwise stated, all vector spaces and linear combinations are over field $\mathbb{R}$ and the orthogonality is with respect to the usual Euclidean inner product. By span$\{\underline{w}_1,\ldots,\underline{w}_k\}$ we mean the vector space generated by vectors $\underline{w}_1,\ldots,\underline{w}_k$ over $\mathbb{R}$, and by dim($\mathcal{W}$) we mean the dimension of vector space $\mathcal{W}$ over its base field $\mathbb{R}$. $u(t)$ is unit step (aka Heaviside step) function defined as $u(t) = 1$ if $t \geq 0$ and $u(t) = 0$ if $t < 0$; $\star$ is the convolution operator. $\mathscr{L} : f(t) \mapsto F(s)$ and $\mathscr{L}^{-1} : F(s) \mapsto f(t)$ denote the unilateral Laplace and inverse Laplace transforms for $t \geq 0$, respectively. Primes of functions of one variable denote the derivatives with respect to the variable, for instance, $y'(x) = \frac{d}{dx}y(x)$, and by $y^{(n)}(x)$ we mean the n-th order derivative of $y(x)$ with respect to variable x.

1. Let $\mathbf{A} = [\underline{a}_1,\ldots,\underline{a}_n] \in \mathbb{R}^{m\times n}$. If rank($\mathbf{A}$) $= m$, which of the following statements is/are true?

   (A) span$\{\underline{a}_1,\ldots,\underline{a}_n\} = \mathbb{R}^m$.

   (B) The linear system $\mathbf{A}\underline{x} = \underline{b}$ with unknown $\underline{x}$ is consistent for all $\underline{b} \in \mathbb{R}^m$.

   (C) The reduced row-echelon form of $\mathbf{A}$ does not have any zero row.

   (D) Each row of $\mathbf{A}$ has a pivot position.

   (E) None of the above is true.

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2. Let $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{n \times n}$ be invertible matrices. Which of the following statements is/are always true?

(A) $\mathbf{M}^{-1}$ is invertible and $(\mathbf{M}^{-1})^{-1} = \mathbf{M}$.

(B) $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$.

(C) $\mathbf{MN} = \mathbf{NM}$.

(D) $\mathbf{M}^\top$ is invertible and $(\mathbf{M}^\top)^{-1} = (\mathbf{M}^{-1})^\top$.

(E) None of the above is true.

3. Let $\mathbf{B} \in \mathbb{R}^{2 \times 2}$ and we know that

$$\mathbf{B}^2 = \frac{1}{8}\begin{bmatrix} 17 & 15 \\ 15 & 17 \end{bmatrix}.$$

Which of the following statements can be true?

(A)

$$\left(\mathbf{B}^{-1}\right)^3 = \frac{1}{16}\begin{bmatrix} 65 & -63 \\ -63 & 65 \end{bmatrix}.$$

(B)

$$\left(\mathbf{B}^{-1}\right)^3 = \frac{1}{16}\begin{bmatrix} -65 & 63 \\ 63 & -65 \end{bmatrix}.$$

(C)

$$\left(\mathbf{B}^{-1}\right)^3 = \frac{1}{16}\begin{bmatrix} -63 & 65 \\ 65 & -63 \end{bmatrix}.$$

(D)

$$\left(\mathbf{B}^{-1}\right)^3 = \frac{1}{16}\begin{bmatrix} 63 & -65 \\ -65 & 63 \end{bmatrix}.$$

(E) None of the above can be true.

4. Which of the following statements is/are false?

(A) If $\mathcal{V}$ is a real vector space of dimension k, then every generating set for $\mathcal{V}$ contains exactly k real vectors.

(B) If $\mathcal{S}$ is a linearly independent set of k vectors from a vector space $\mathcal{V}$ of dimension k, then $\mathcal{S}$ is a basis for $\mathcal{V}$.

(C) If $\mathcal{V}$ is a subspace of $\mathcal{W}$, then the dimension of $\mathcal{V}$ is less than or equal to the dimension of $\mathcal{W}$.

(D) If T is a linear operator on $\mathbb{R}^n$, $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ is a basis for $\mathbb{R}^n$, $\mathbf{B} = [\underline{b}_1, \ldots, \underline{b}_n]$, and $\mathbf{A}$ is the standard matrix of T, then the matrix representation of T with respect to $\mathcal{B}$ is $[T]_{\mathcal{B}} = \mathbf{BAB}^{-1}$.

(E) All of the above are true.

5. On the two-dimensional xy-plane, let $T(\underline{m})$ be the linear transformation that will rotate the vector $\underline{m} = [m_1, m_2]^\top$ by 60 degrees counterclockwise, and let $U(\underline{m})$ be the linear transformation that will generate the reflection of $\underline{m}$ about the x-axis. Which of the following statements is/are true regarding $U(T(\underline{m})) = [u_1, u_2]^\top$ given $\underline{m} = [1, 1]^\top$?

(A) $u_1 + u_2 = 1$.

(B) $u_2 - u_1 = 1$.

(C) $2u_1u_2 = 1$.

(D) $u_1 < 1$.

(E) None of the above is true.

6. For a symmetric and positive semi-definite matrix $\mathbf{A} = [a_{i,j}] \in \mathbb{R}^{n\times n}$, which of the following statements is/are true?

(A) $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j} \geq 0$.

(B) $\left(\frac{1}{n}\text{tr}(\mathbf{A})\right)^n < \det(\mathbf{A})$.

(C) $|\underline{x}^\top \mathbf{A}\underline{y}| \leq \sqrt{(\underline{x}^\top \mathbf{A}\underline{x})\,(\underline{y}^\top \mathbf{A}\underline{y})}$ for all $\underline{x}, \underline{y} \in \mathbb{R}^n$.

(D) If no eigenvalues of $\mathbf{A}$ are in the interval $[p, q]$, then $(\mathbf{A} - p\,\mathbf{I}_n)(\mathbf{A} - q\,\mathbf{I}_n)$ is positive definite.

(E) None of the above is true.

7. Let $\mathcal{M}$ and $\mathcal{N}$ be two non-trivial subspaces of $\mathbb{R}^n$ with sum $\mathcal{M}+\mathcal{N} = \{\underline{y}+\underline{z} : \underline{y} \in \mathcal{M}, \underline{z} \in \mathcal{N}\} = \mathbb{R}^n$. Which of the following statements is/are true?

(A) $\dim(\mathcal{M}) + \dim(\mathcal{N}) = n$.

(B) For any $\underline{x} \in \mathbb{R}^n$, there exists a unique pair $\underline{y} \in \mathcal{M}$ and $\underline{z} \in \mathcal{N}$ such that $\underline{x} = \underline{y} + \underline{z}$.

(C) If $\mathcal{B}_\mathcal{M}$ and $\mathcal{B}_\mathcal{N}$ are respectively bases for $\mathcal{M}$ and $\mathcal{N}$, then $\mathcal{B}_\mathcal{M} \cup \mathcal{B}_\mathcal{N}$ is a basis for $\mathbb{R}^n$.

(D) Let $\mathbf{P}$ be the orthogonal projection matrix from $\mathbb{R}^n$ onto $\mathcal{M}$. Then $\underline{x}^\top \mathbf{P}\underline{x} = 0$ whenever $\underline{x} \in \mathcal{N}$.

(E) None of the above is true.

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8. Let $\mathcal{U}$ and $\mathcal{V}$ be two finite-dimensional real inner product spaces with Euclidean inner products $\langle \cdot, \cdot \rangle_{\mathcal{U}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ and with ordered orthonormal bases $\mathcal{B}_{\mathcal{U}} = \{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ and $\mathcal{B}_{\mathcal{V}} = \{\underline{v}_1, \underline{v}_2\}$, respectively. Let $T : \mathcal{U} \to \mathcal{V}$ be a linear transformation, and the matrix representation of T with respect to $\mathcal{B}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{V}}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Which of the following statements is/are true?

(A) T is one-to-one and onto.

(B) If $\underline{u} = \underline{u}_1 - \underline{u}_3$, then $\sqrt{\langle T(\underline{u}), T(\underline{u}) \rangle_{\mathcal{V}}} = \sqrt{2}$.

(C) For all $\underline{u} \in \mathcal{U}$ satisfying $T(\underline{u}) = \underline{v}_1 + 3\underline{v}_2$, the minimum value of $\sqrt{\langle \underline{u}, \underline{u} \rangle_{\mathcal{U}}}$ is equal to $\sqrt{5}$.

(D) The element $\underline{u}$ that achieves the minimal value of $\sqrt{\langle \underline{u}, \underline{u} \rangle_{\mathcal{U}}}$ in statement (C) is $\underline{u}_1 + 2\underline{u}_3$.

(E) None of the above is true.

9. For a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ with $\text{rank}(\mathbf{A}) = k < n$, write $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q} \in \mathbb{R}^{n \times k}$ has orthonormal columns and $\mathbf{R} = [r_{i,j}] \in \mathbb{R}^{k \times n}$ satisfying $r_{i,j} = 0$ for all $i = 2, \ldots, k$ and $j = 1, \ldots, i-1$. Which of the following statements is/are always true?

(A) The row space of $\mathbf{R}$ equals the row space of $\mathbf{A}$.

(B) The space spanned by the first $k+1$ columns of $\mathbf{A}$ has dimension equal to k.

(C) Given $\underline{b} \in \mathbb{R}^n$, the vector $\underline{x} \in \mathbb{R}^n$ that minimizes $\|\mathbf{A}\underline{x} - \underline{b}\|$ satisfies $\mathbf{A}\underline{x} = \mathbf{Q}\mathbf{Q}^\top \underline{b}$.

(D) The vector $\underline{x}$ in the above statement (C) is unique.

(E) None of the above is true.

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10. A square matrix $\mathbf{A} = [a_{i,j}] \in \mathbb{R}^{3\times 3}$ with $a_{i,j} \neq 0$ for all i, j has eigenvalues 0, 1 and 3. Which of the following statements is/are true?

(A) $\mathbf{A}$ is diagonalizable.

(B) $\text{rank}(\mathbf{A}\mathbf{A}^\top) = 1$.

(C) The linear equation $(\mathbf{A} + 2\mathbf{I}_3)\underline{x} = \underline{b}$ with unknown $\underline{x}$ is consistent for any $\underline{b} \in \mathbb{R}^3$.

(D) If $\mathbf{A}$ is symmetric, then $\underline{y}^\top \mathbf{A}\underline{x}$ defines an inner product for all $\underline{x}, \underline{y} \in \mathbb{R}^3$.

(E) None of the above is true.

11. Consider the following differential equation:

$$(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)\,dy = 0.$$

Which of the following statements is/are true?

(A) The normal form of this equation is not separable.

(B) This equation is not exact.

(C) There exists an integrating factor that depends only on x.

(D) There exists an integrating factor that depends only on y.

(E) None of the above is true.

12. Continue from Problem 11. Suppose $(x, y) = (0, 1)$ is on the solution curve of the differential equation. Which of the following statements is/are true?

(A) $(x, y) = (1, 0)$ is on the solution curve.

(B) $(x, y) = (2, 2)$ is on the solution curve.

(C) $(x, y) = (\frac{1}{2}, \frac{1+\sqrt{2}}{2})$ is on the solution curve.

(D) $(x, y) = (4, 2)$ is on the solution curve.

(E) None of the above is true.

13. Consider the following differential equation:

$$y''(x) + y'(x) + (1 + x - x^2)y(x) = 0.$$

Which of the following statements is/are true?

(A) This is a nonlinear equation.

(B) There exists a trivial solution $y(x) = 0$ for all x.

(C) There exists a nontrivial solution for this equation.

(D) For this equation, there exist three particular solutions that are linearly independent over $\mathbb{R}$.

(E) None of the above is true.

14. Continue from Problem 13. Determine the particular solution $y(x)$ satisfying $y(0) = 1$ and $y'(0) = 0$. Which of the following statements is/are true?

(A) $y''(0) = -1$.

(B) $\lim_{x\to\infty} y(x) = 0$.

(C) $y(1) = e^{-1}$.

(D) $y(2) = e^{-2}$.

(E) None of the above is true.

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15. Consider the following system of differential equations:

$$x'(t) - 4x(t) + y''(t) = t^2$$
$$x'(t) + x(t) + y'(t) = 0$$

Suppose $x(0) = x'(0) = y(0) = 0$. Which of the following statements is/are true?

(A) $y''(t) + 4y(t) + 1 = 0$.

(B) $x''(t) + 4x(t) + t^2 = 0$.

(C) $x^{(4)}(0) = -2$.

(D) $y^{(4)}(0) = \pi$.

(E) None of the above is true.

16. Consider the following second order differential equation

$$y''(x) + (x-2)y'(x) + y(x) = x,$$

where the function $y(x)$ satisfies the conditions $y(2) = 2$ and $y'(2) = 1$. Let

$$y(x) = (x-2)^r \sum_{n=0}^{\infty} a_n (x-2)^n$$

with $r, a_n \in \mathbb{R}$ and $a_0 \neq 0$ be the series solution to the above differential equation. Which of the following statements is/are true regarding the values of r and a_n?

(A) $r = 0$.

(B) $a_2 = \frac{1}{2}$.

(C) $a_3 = -\frac{1}{6}$.

(D) $a_4 = \frac{1}{30}$.

(E) None of the above is true.

17. Consider the following differential equation defined for all $t \in \mathbb{R}$

$$tx''(t) - 2x'(t) - tx(t) = g(t)$$

with initial conditions $x(0) = 0$ and $x(1) = 1$, where $g(t) = f(t) \star f(t)$ and $f(t) = \sqrt{t-1}u(t-1)$. Which of the following statements is/are true regarding the values of $F(s) = \mathcal{L}\{f(t)\}$ and the function $g(t)$?

(A) $F(s)$ exists for all $s \in \mathbb{R}$ with $s \geq 1$.

(B) $F(1) > 1$.

(C) $g(1.5) > 0$.

(D) $g(4) > 1$.

(E) None of the above is true.

18. Continue from Problem 17. Which of the following statements is/are true regarding the solution $x(t)$ to the differential equation?

(A) $x(t)$ is an even function for $|t| \leq \frac{1}{2}$.

(B) $x(-1) = 1$.

(C) $x'(0) = 0$.

(D) $x'(1) < 2$.

(E) None of the above is true.

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19. Consider the following boundary value problem for the bivariate function $v(x,t)$

$$\frac{\partial}{\partial t}v(x,t) = -\frac{\partial^2}{\partial x^2}v(x,t)$$

that is defined for $x \in [-\pi, \pi]$ and $t \geq 0$ and satisfies the following conditions

$$v(-\pi,t) = v(\pi,t), \qquad \text{for all } t > 0$$

$$\frac{\partial}{\partial x}v(x,t)\bigg|_{x=-\pi} = \frac{\partial}{\partial x}v(x,t)\bigg|_{x=\pi}, \qquad \text{for all } t > 0$$

$$v(x,0) = \begin{cases} 2\sin(2x), & \text{if } x \in [0,\pi] \\ \sin(4x), & \text{if } x \in [-\pi,0] \end{cases}$$

The solution $v(x,t)$ can be represented in the following form

$$v(x,t) = \sum_{n=0}^{\infty} e^{d_n t}\left[a_n \cos(f_n x) + b_n \sin(f_n x)\right]$$

for some $a_n, b_n, d_n, f_n \in \mathbb{R}$ with $0 \leq f_0 < f_1 < \cdots$ and $|a_n| + |b_n| > 0$ for all $n = 0, 1, \ldots$ Which of the following statements is/are true regarding the values of a_n and b_n in the representation of solution $v(x,t)$?

(A) $a_0 \geq 1$ and $b_0 \leq 0$.

(B) $a_1 \leq 1$ and $b_1 \geq 1$.

(C) $a_2 \leq 0$ and $b_2 \geq 0$.

(D) $a_3 \geq 0$ and $b_3 \leq 1$.

(E) None of the above is true.

20. Continue from Problem 19. Which of the following statements is/are true regarding the values of d_n and f_n in the representation of solution $v(x,t)$?

(A) $f_0 = 0$.

(B) $f_1 = 2$.

(C) $d_2 = -9$.

(D) $d_3 = 25$.

(E) None of the above is true.