科目：工程數學C(3006)　　　校系所組：中央大學電機工程學系(電子組)
交通大學電子研究所(甲組、乙A組、乙B組)
交通大學電控工程研究所
交通大學電機工程學系(甲組)
交通大學電信工程研究所（乙組）
交通大學生醫工程研究所(乙組)
清華大學電機工程學系(甲組)
清華大學光電工程研究所
清華大學電子工程研究所
清華大學工程與系統科學系（丁組）

- 本測驗試題為複選題（答案可能有一個或多個），請選出所有正確或最適當的答案，並請用2B鉛筆作答於答案卡。
- 共二十題，每題完全答對得五分，答錯不倒扣。
- In the following questions, $\delta(t)$ is the Dirac delta function, $u(t)$ is unit-step function, $\star$ is the convolution operator, $\mathcal{L}: f(t) \mapsto F(s)$ and $\mathcal{L}^{-1}: F(s) \mapsto f(t)$ denote the unilateral Laplace and inverse Laplace transforms for $t \geq 0$, respectively, boldface letters such as $\boldsymbol{a}, \boldsymbol{b}$, etc. denote vectors of proper length, A^{T} means the transpose of matrix A, and I is the identity matrix of proper size.

一、 Suppose that B_1 and B_2 are square invertible matrices. Let $\mathbf{0}$ denote an all-zero matrix of proper size. Define

$$B = \begin{bmatrix} B_1 & \mathbf{0} \\ \mathbf{0} & B_2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} B_1^{-1} & \mathbf{0} \\ \mathbf{0} & B_2^{-1} \end{bmatrix}.$$

Denote by P a permutation matrix of the same size as B and C. Which of the following statements are true?

(A) $BC = P^{\mathrm{T}}P$

(B) $P^{\mathrm{T}}P = PP^{\mathrm{T}}$.

(C) $B^{\mathrm{T}}CBPC^{-1}B^{-1}P^{\mathrm{T}}C^{\mathrm{T}} = I$

(D) There exists an integer $k \neq 1$ such that $P^k = P$.

(E) None of the above are true.

二、 Using the forward elimination process with possible row exchanges to produce an upper triangular matrix U, which of the following statements are true?

(A) When performing forward elimination on matrix $B = \begin{bmatrix} B_1 & \mathbf{0} \\ \mathbf{0} & B_2 \end{bmatrix}$ to produce an upper triangular matrix U, where B_1 and B_2 are square matrices and $\mathbf{0}$ denotes an all-zero matrix of proper size, U can be made to be equal to $U = \begin{bmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{bmatrix}$, where U_1 and U_2 are respectively the resulting forward elimination upper triangular outputs due to inputs B_1 and B_2.

(B) Suppose that $P_1B_1 = L_1U_1$ and $P_2B_2 = L_2U_2$, where P_1 and P_2 are permutation matrices, L_1 and L_2 are square lower triangular matrices, and U_1 and U_2 are square upper triangular matrices. Then,

$$\begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & P_2 \end{bmatrix} \begin{bmatrix} B_1 & \mathbf{0} \\ \mathbf{0} & B_2 \end{bmatrix} = \begin{bmatrix} L_1 & \mathbf{0} \\ \mathbf{0} & L_2 \end{bmatrix} \begin{bmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{bmatrix}.$$

(C) To perform forward elimination on a matrix $F = \begin{bmatrix} A & \mathbf{0} \\ C & D \end{bmatrix}$ to produce an upper triangular matrix U, where A, C and D are square invertible matrices, we can first do block-based forward elimination to obtain $G = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix}$; then perform forward elimination respectively on A and D to obtain upper triangular U_A and U_D. The desired U is thus given by $\begin{bmatrix} U_A & \mathbf{0} \\ \mathbf{0} & U_D \end{bmatrix}$.

注意：背面有試題

科目：工程數學C(3006)　　校系所組：中央大學電機工程學系(電子組)
交通大學電子研究所(甲組、乙A組、乙B組)
交通大學電控工程研究所
交通大學電機工程學系(甲組)
交通大學電信工程研究所（乙組）
交通大學生醫工程研究所(乙組)
清華大學電機工程學系(甲組)
清華大學光電工程研究所
清華大學電子工程研究所
清華大學工程與系統科學系（丁組）

(D) To perform forward elimination on a matrix $F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ to produce an upper triangular matrix U, where A, B, C and D are square invertible matrices, U can be made to be equal to $\begin{bmatrix} U_A & \mathbf{0} \\ \mathbf{0} & U_D \end{bmatrix}$, where U_A and U_D are respectively the forward elimination upper triangular outputs due to inputs A and D.

(E) None of the above are true.

三、 Let $A \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{R}^n$ be nonzero matrix and nonzero vector, respectively. Which of the following sets are subspaces of $\mathbb{R}^n$?

(A) $\{\boldsymbol{x} \in \mathbb{R}^n \,|A\boldsymbol{x} = \boldsymbol{b}\}$.

(B) $W_1 \cap W_2$, where W_1 and W_2 are two subspaces of $\mathbb{R}^n$.

(C) $\{\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n \,|\sum_{i=1}^n x_i = 0\}$.

(D) $\{\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}} \,|\boldsymbol{x}'(t) = A\boldsymbol{x}(t) + \boldsymbol{b}\}$.

(E) None of the above are true.

四、 Which of the following statements are true?

(A) Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$, $k < n$, be linearly independent unit vectors in $\mathbb{R}^n$ and $U = \begin{bmatrix} \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_k \end{bmatrix} \in \mathbb{R}^{n\times k}$. Then, $I_n - UU^{\mathrm{T}}$ is a projection matrix that projects a vector onto the column space of U, where I_n is an identity matrix of size $n \times n$.

(B) Let $A \in \mathbb{R}^{n\times k}$, $k < n$ and $\mathrm{rank}(A) = k$. Then $A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is the projection matrix that projects a vector onto the column space of A.

(C) Let $A \in \mathbb{R}^{n\times k}$, $n < k$ and $\mathrm{rank}(A) = n$. Then $\boldsymbol{x} = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}\boldsymbol{b}$ is the solution of $A\boldsymbol{x} = \boldsymbol{b}$ having minimum Euclidean norm.

(D) All the orthogonal matrices of $\mathbb{R}^{2\times 2}$ can be expressed either in the form of $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ or $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$.

(E) None of the above are true.

五、 Let $A = QR$ be the QR factorization of A, where $A = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \end{bmatrix} \in \mathbb{R}^{n\times n}$, and $Q = \begin{bmatrix} \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_n \end{bmatrix} \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, and $R = \begin{bmatrix} r_{i,j} \end{bmatrix} \in \mathbb{R}^{n\times n}$ is an upper triangular matrix. Which of the following statements are true?

(A) $\boldsymbol{a}_k$ and $\boldsymbol{q}_k$ are linearly dependent for any $k \le n$.

(B) $\mathrm{span}\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k\} = \mathrm{span}\{\boldsymbol{q}_1, \cdots, \boldsymbol{q}_k\}$ for any $k \le n$.

(C) $\{\boldsymbol{q}_i \,|r_{i,i} \ne 0,\ i = 1, 2, \ldots, n\}$ is an orthonormal basis of the column space of A.

(D) If $r_{k,k} = 0$, then the vectors $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k\}$ are linearly dependent.

(E) None of the above are true.

六、 Consider a 4×4 real matrix A with three different eigenvalues 0, 1, 2. Which of the following statements are true?

注意：背面有試題

科目：工程數學 C(3006)　　校系所組：中央大學電機工程學系(電子組)
交通大學電子研究所(甲組、乙A組、乙B組)
交通大學電控工程研究所
交通大學電機工程學系(甲組)
交通大學電信工程研究所（乙組）
交通大學生醫工程研究所(乙組)
清華大學電機工程學系(甲組)
清華大學光電工程研究所
清華大學電子工程研究所
清華大學工程與系統科學系（丁組）

(A) The determinant of A is 0.
(B) There are three linearly independent eigenvectors.
(C) The rank of A is 2.
(D) The trace of A is 3.
(E) None of the above are true.

七、 Consider two similar real matrices A and B. Which of the following statements are true?

(A) A and B have the same set of eigenvalues.
(B) A and B have the same set of eigenvectors.
(C) A and B have the same null space.
(D) A and B have the same rank.
(E) None of the above are true.

八、 Consider an $m \times n$ real matrix A with linearly independent columns, and $m > n$. Which of the following statements are true?

(A) $A^{\mathrm{T}}A$ is positive definite.
(B) AA^{T} is positive definite.
(C) The column space of A is spanned by all the eigenvectors of AA^{T}.
(D) The row space of A is spanned by all the eigenvectors of $A^{\mathrm{T}}A$.
(E) None of the above are true.

九、 Which of the following statements are true?

(A) Let $A = \begin{bmatrix} 1+\alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & 1+\alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \cdots & 1+\alpha_n \end{bmatrix}$. Then, $\det(A) = 1 + n\sum_{i=1}^{n} \alpha_n$.
(B) Let A be a square matrix, and $\boldsymbol{c}$ and $\boldsymbol{d}$ be two column vectors. If $A\boldsymbol{x} = \boldsymbol{c}$, then $\det(A + \boldsymbol{c}\boldsymbol{d}^{\mathrm{T}}) = \det(A)(1 + \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x})$.
(C) Consider the 4×4 orthogonal projection matrix $Q = I - \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$, where $\boldsymbol{u} \in \mathbb{R}^4$ and I is the 4×4 identity matrix. Then $\det(Q) + \operatorname{rank}(Q) = 4$.
(D) One can construct a 3×3 Hermitian matrix A with complex-valued entries such that $\det(A) = 1 + i$, where $i = \sqrt{-1}$.
(E) None of the above are true.

十、 Which of the following statements are true?

(A) Let T be a linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^m$. Then T is one-to-one.
(B) Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ can be linearly dependent even if $\{T\boldsymbol{v}_1, \ldots, T\boldsymbol{v}_k\}$ is linearly independent.

注意：背面有試題

科目：工程數學C(3006)　　校系所組：中央大學電機工程學系(電子組)
交通大學電子研究所(甲組、乙A組、乙B組)
交通大學電控工程研究所
交通大學電機工程學系(甲組)
交通大學電信工程研究所(乙組)
交通大學生醫工程研究所(乙組)
清華大學電機工程學系(甲組)
清華大學光電工程研究所
清華大學電子工程研究所
清華大學工程與系統科學系(丁組)

(C) Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, and let $S \subset \mathbb{R}^n$ be a subspace such that $Ts \in S$ for all $s \in S$. Then $\mathtt{dim}(S) \in \{0, 1, n\}$, where $\mathtt{dim}(S)$ denotes the dimension of S.

(D) Let $A \in \mathbb{R}^{m\times n}$ with $m > n > 3$. Then $\mathtt{rank}(AA^{\mathsf{T}}) < \mathtt{rank}(A)$.

(E) None of the above are true.

十一、 Consider the first order differential equation $(4xy + 1)\,dy + y^2 dx = 0$. Which of the following statements are true?

(A) $x = y^{-1} + Cy^{-4}$ for some constant C is the general solution.

(B) $3xy^4 + y^3 = C$, where $y \neq 0$ and C is some constant, is an implicit solution.

(C) $y(x) = 0$ is a solution.

(D) If the solution curve passes through the point $(0, 3)$ in the $x - y$ plane, then it also passes through the point $(7, 1)$.

(E) None of the above are true.

十二、 Given one solution $y_1(x) = e^x$ to the homogeneous second order linear differential equation $(x+1)y''(x) - (x+2)y'(x) + y(x) = 0$ with $x > -1$, the second linearly independent solution $y_2(x)$ takes the form of $y_2(x) = v(x)y_1(x)$. Which of the following statements are true?

(A) The function $v(x)$ satisfies $(x+1)v''(x) - (x+2)v'(x) = 0$.

(B) The function $v(x)$ satisfies $(x+1)v''(x) + xv'(x) = 0$.

(C) $v(x) = xe^x$.

(D) $v(x) = (2+x)e^{-x}$.

(E) None of the above are true.

十三、 Solve the third order differential equation $y'''(x) - \frac{3}{x}y''(x) + \frac{6}{x^2}y'(x) - \frac{6}{x^3}y(x) = \frac{1}{x^4}$ with $x > 0$. Which of the following statements are true?

(A) $y(x) = -\frac{1}{24}x^{-1}$ is a solution.

(B) $y(x) = \frac{1}{24}x$ is a solution.

(C) $y(x) = \frac{1}{24}(x - x^{-1})$ is a solution.

(D) If the solution $y(x)$ satisfies $y(1) = 0$, $y'(1) = \frac{1}{6}$, and $y''(1) = \frac{1}{6}$, then $y(2) = \frac{1}{16}$.

(E) None of the above are true.

十四、 Consider the system of linear differential equations

$$x'(t) = Ax(t), \quad \text{where } A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}.$$

Which of the following statements are true?

(A) $x(t) = \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix}$ is a solution.

(B) $x(t) = \begin{bmatrix} \cos(t) - 2\sin(t) \\ \sin(t) \end{bmatrix}$ is a solution.

注意：背面有試題

科目：工程數學 C(3006)　校系所組：中央大學電機工程學系(電子組)
交通大學電子研究所(甲組、乙A組、乙B組)
交通大學電控工程研究所
交通大學電機工程學系(甲組)
交通大學電信工程研究所（乙組）
交通大學生醫工程研究所(乙組)
清華大學電機工程學系(甲組)
清華大學光電工程研究所
清華大學電子工程研究所
清華大學工程與系統科學系（丁組）

(C) $e^{\pi A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

(D) $e^{\frac{\pi}{2}A} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$.

(E) None of the above are true.

十五、 Which of the following statements are true?

(A) $\mathcal{L}\left\{\int_0^t \sin(a(t-\tau))\sin(a\tau)\, d\tau\right\} = \frac{1}{(s^2+a^2)^2}$.

(B) $(\delta \star f)(t) \neq f(t)$.

(C) $(u \star f)(t) = f(t)$, where $u(t)$ is the unit-step function.

(D) If $y'(t) + 2y(t) + \int_0^t y(\tau)\, d\tau = u(t-1)$ and $y(0) = 0$, then $Y(s) = \mathcal{L}\{y(t)\} = \frac{e^{-s}}{(s+1)^2}$.

(E) None of the above are true.

十六、 Let $x(t)$ be the solution of the initial value problem $x''(t) + p_0 x'(t) + q_0 x(t) = f(t)$, $t \geq 0$, $x(0) = a$ and $x'(0) = b$. Let $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{x(t)\} = X(s)$. If $X(s) = \frac{2s+1}{s^2+2s+1} + \frac{F(s)}{s^2+2s+1}$, then which of the following statements are true?

(A) $p_0 + q_0 = -3$.

(B) $a + b = -1$.

(C) If $f(t) = 0$, then $x(t) = 2e^{-t} - te^{-t}$.

(D) If $f(t) = \delta(t-3)$, then $\mathcal{L}^{-1}\{\frac{F(s)}{s^2+2s+1}\} = u(t-3)te^{-t}$.

(E) None of the above are true.

十七、 Consider the following non-homogeneous linear system

$$\boldsymbol{x}'(t) = A\boldsymbol{x}(t) + \boldsymbol{f}(t), \text{ where } A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \text{ and } \boldsymbol{f}(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} \qquad (*)$$

Given that $\lambda_1 = 1$ and $\lambda_2 = 1 + 2i$ are eigenvalues for A and that $\begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue λ_2, which of the following statements are true?

(A) $\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} e^t$ is a solution to $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$.

(B) $e^{At} = \begin{bmatrix} 2e^t & 0 & 0 \\ -3e^t & e^t\cos(2t) & e^t\sin(2t) \\ 2e^t & e^t\sin(2t) & -e^t\cos(2t) \end{bmatrix}$.

(C) The system (*) has a particular solution $\boldsymbol{x}_p(t) = \boldsymbol{b}e^{2t}$ with $(A - 2I)\boldsymbol{b} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$.

注意：背面有試題

科目：工程數學C(3006)　　校系所組：中央大學電機工程學系(電子組)
交通大學電子研究所(甲組、乙A組、乙B組)
交通大學電控工程研究所
交通大學電機工程學系(甲組)
交通大學電信工程研究所（乙組）
交通大學生醫工程研究所(乙組)
清華大學電機工程學系(甲組)
清華大學光電工程研究所
清華大學電子工程研究所
清華大學工程與系統科學系（丁組）

(D) $x_p(t) = \begin{bmatrix} -1 \\ \frac{4}{5} \\ -\frac{7}{5} \end{bmatrix} e^{2t}$ is a particular solution to the system (*).

(E) None of the above are true.

十八、 Let $f(t) = \sum_{n=-\infty}^{\infty} g(t-n)$, where $g(t) = t^2\left[u(t) - u(t-2)\right]$. Represent $f(t)$ in terms of Fourier series as $\tilde{f}(t) = \sum_{n\geq 0} a_n \cos(n\pi t) + b_n \sin(n\pi t)$. Which of the following statements are true?

(A) $f(t)$ is a periodic function with minimum period 2.

(B) $b_n = -\frac{4(1+(-1)^n)}{n\pi}$ for $n \geq 1$.

(C) $b_n = -\frac{4(1-(-1)^n)}{n\pi}$ for $n \geq 1$.

(D) $\tilde{f}(t=1) = 3$.

(E) None of the above are true.

十九、 Consider the following boundary value problem for the bivariate function $g(x,t)$ with $10 < x < 20$

$$\frac{\partial g(x,t)}{\partial t} = 5\frac{\partial^2 g(x,t)}{\partial x^2} \quad \text{for } 10 < x < 20 \text{ and } t > 0$$

$$g(10,t) = g(20,t) = 0 \quad \text{for } t > 0$$

Assuming $g(x,t) = X(x)T(t)$ is separable, which of the following statements are true?

(A) The function $X(x)$ satisfies $X''(x) + X(x) = 0$ with $X(10) = X(20) = 0$.

(B) Solutions to $X(x)$ take the form of $(-1)^n \sin\left(\frac{n\pi}{10}x\right)$ for $n = 1, 2, \ldots$

(C) Assuming $X(x) = \sin\left(\frac{n\pi}{20}x\right)$ for some positive integer n, the corresponding $T(t)$ satisfies $T'(t) + \frac{n^2\pi^2}{80}T(t) = 0$.

(D) Assuming $X(x) = \sin\left(\frac{n^2\pi}{10}x\right)$ for some positive integer n, the corresponding $T(t)$ satisfies $T'(t) + \frac{n\pi}{10}T(t) = 0$.

(E) None of the above are true.

二十、 Continued from Question 十九, we further assume that $g(x,0) = x$ for $10 < x < 20$ and that the solution $g(x,t)$ takes the form of

$$g(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$$

for some constants c_n. Which of the following statements are true?

(A) $c_n = \frac{20((-1)^n - 2)}{n\pi}$, $X_n(x) = \sin\left(\frac{n\pi}{10}x\right)$ and $T_n(t) = \exp\left(-\frac{n^2\pi^2}{20}t\right)$.

(B) $c_n = \frac{20(1-2(-1)^n)}{n\pi}$, $X_n(x) = \sin\left(\frac{n\pi}{10}x\right)$ and $T_n(t) = \exp\left(-\frac{n^2\pi^2}{20}t\right)$.

(C) $\sum_{n=1}^{\infty} c_n X_n(20) T_n(0) = 20$.

(D) $\sum_{n=1}^{\infty} c_n X_n(10) T_n(0) = 0$.

(E) None of the above are true.