1 (20%) Find the general solutions or the particular solutions, which satisfy the given initial conditions, for the following differential equations:

(a) $y'' + 3.2y' + 2.56y = 0$.

(b) $2y'' - 9y' = 0$.

(c) $y'' + 0.4y' + 0.29y = 0$, $\quad y(0) = 1$, $\quad y'(0) = -1.2$.

(d) $y'' - k^2 y = 0 \quad (k \neq 0)$, $\quad y(0) = 1$, $\quad y'(0) = 1$.

2 (10%) It is obvious that e^x satisfies the differential equation

$$y'' - y = 0.$$

Then, by using the method of REDUCTION OF ORDER, please prove that e^{-x} is another basis for this equation.

3 (10%) Solve $y' = -2xy$ by the method of power series.

4 (15%) If $f(t)$ has the Laplace transform of $F(s)$ and the operator $\mathcal{L}\{\ \}$ denotes the Laplace transformation, please prove the followings:

(a) $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$, here u is the unit step function.

(b) $\mathcal{L}\{\delta(t-a)\} = e^{-as}$, here δ is the Dirac's delta function.

(c) $(\delta * f)(t) = f(t)$, here $*$ means the convolution operator. (*Hint*: Use the convolution theorem.)

5 (10%) For an infinite bar one-dimensional heat equation $\dfrac{\partial u(x,t)}{\partial t} = c^2 \dfrac{\partial^2 u(x,t)}{\partial x^2}$, with the initial condition $u(x,0) = f(x)$, has an solution of the error function form

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2cz\sqrt{t})e^{-z^2}\,dz.$$

If $f(x) = 1$ when $x > 0$ and $f(x) = 0$ when $x < 0$, please show that

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\frac{x}{2c\sqrt{t}}}^{\infty} e^{-z^2}\,dz \qquad \text{for } t > 0.$$

注意：背面有試題

6 (15%) The matrices P1= $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, P2= $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and P3= $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ occur in the quantum mechanical theory of electron spin and are called *Pauli spin matrices*, in honor of the physicist Wolfgang Pauli (1900-1958). Here i is the imaginary unit.

(a) Verify that they all have eigenvalues 1 and –1.

(b) Determine all 2x2 matrices with complex entries having the two eigenvalues 1 and –1.

(c) Show that any rank 2 matrix M can be represented as a sum of a 2x2 unit matrix and these Puli spin matrices.

7 (20%) A general form of the solution within the spatial interval $(0,L)$ for the one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

with two boundary conditions: $u(0,t) = 0$ and $u(L,t) = 0$ for all t

and two initial conditions: $u(x,0) = f(x)$ and $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos\frac{cn\pi}{L}t + B_n^* \sin\frac{cn\pi}{L}t)\sin\frac{n\pi}{L}x$$

where $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$ and $B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$.

Now please write the explicit solution, i.e. don't use the notation of Σ as the final solution and write down explicitly the non-zero u_n's up to four terms, of the one-dimensional wave equation corresponding to the triangular initial deflection

$$f(x) = \begin{cases} 2kx/L & if \quad 0 < x < L/2 \\ 2k(L-x)/L & if \quad L/2 < x < L \end{cases}$$

and initial velocity zero. (*Hint*: The coefficient b_n of the Fourier Sine Series for the odd periodic extension of $f(x)$ is $b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$.)