國立中央大學八十九學年度轉學生入學試題卷

Denote by $\mathbb{R}$ the set of all real numbers, $[a,b]$ be a bounded closed interval in $\mathbb{R}$.

1. (10%) Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $x \neq y$. Define the line segment

$$L(x,y) = \{tx + (1-t)y : t \in [0,1]\}.$$

Prove that the line segment $L(x,y)$ is a connected set in $\mathbb{R}^n$.

2. (15%) (a) Let $f : [a,b] \to \mathbb{R}$ be bounded on [a,b]. Give a definition for the function f to be Riemann integrable on $[a,b]$.

(b) Prove that a monotone function f on $[a,b]$ is Riemann integrable.

3. (20%) Let $f : [a,b] \to \mathbb{R}$ be increasing on $[a,b]$. Prove that f has at most countably many points of discontinuity on $[a,b]$.

4. (20%) (a) Find $\lim_{n\to\infty} (1+\frac{1}{n})^{n^2} e^{-n}$.

(b) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} (1+\frac{1}{n})^{n^2} (x-1)^n.$$

5. (20%) (a) Let $D \subset \mathbb{R}$ and $f_n : D \to \mathbb{R}$, $f : D \to \mathbb{R}$. Give the definitions that "f_n converges to f pointwise on D" and that "f_n converges to f uniformly on D".

(b) Let $f_n = x^n$ for $x \in [0,1)$. Find $f : [0,1) \to \mathbb{R}$ such that $f_n \to f$ pointwise on $[0,1)$. Does f_n converge uniformly on $[0,1)$? Prove your answer.

Note: You may use the inequality $(1+\theta)^n \geq 1 + n\theta$ for all positive n and $\theta \geq -1$.

6. (15%) (a) Let $D \subset \mathbb{R}^n$ and $f : D \to \mathbb{R}^m$. For $x \in D$ give the definition that f is differentiable at x.

(b) Let

$$\begin{cases} f(x,y) = \dfrac{x^2 y}{x^2+y^2} & \text{for } (x,y) \neq (0,0), \\ f(0,0) = 0. \end{cases}$$

Prove that f is not differentiabe at $(0,0)$.